ON THE CONSTRUCTION OF ORTHOGONAL F(2k,2) SQUARES*

by

D. A. Anderson†, W. T. Federer, and E. Seiden‡

BU-500-M February, 1974

Abstract

In this paper we present three basic techniques for constructing sets of orthogonal $F(n;2)$ squares where $n = 2k$ is even. A case of special interest is $n = 6$ since there are no orthogonal Latin squares of order 6. The first construction is a method of differences which produces a set of eight orthogonal $F(6;2)$ squares. It is shown that this set cannot be extended. The second construction is based on balanced incomplete block designs and produces a set of five orthogonal $F(6;2)$ squares of which four can be embedded in the set of eight constructed by differences. The last construction is for the case where $n + 1 = 4t + 3$ is a prime. It is a difference composition method reducing $(n + 1) \times (n + 1)$ Latin squares to $n \times n$ $F(n;2)$ squares. The construction produces sequences of F squares in which squares adjacent in the sequence are orthogonal.

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* Paper No. BU-500-M in the Biometrics Unit Mimeo Series, Cornell University.

† On leave from the University of Wyoming.

‡ Michigan State University.

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1. Introduction

Let A be an $n \times n$ matrix with elements from the set $\{a_1, a_2, \cdots, a_m\}$. If the element a_k occurs exactly λ_k times, $k = 1, 2, \cdots, m$, in each row and each column, A is called a frequency square, F square, with frequency vector $(\lambda_1, \lambda_2, \cdots, \lambda_k)$. We denote A by $F(n;\lambda_1, \lambda_2, \cdots, \lambda_m)$, and if $n = \lambda m$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_m$, we denote A by $F(n;\lambda)$.

Suppose A is an $F(n;\lambda_1, \lambda_2, \cdots, \lambda_{m_1})$ F square and B is an $F(n;\mu_1, \mu_2, \cdots, \mu_{m_2})$ F square. Then A and B are said to be orthogonal if when A and B are superimposed a_i occurs exactly $\lambda_i\mu_j$ times with b_j, $i = 1, 2, \cdots, m_1$, $b = 1, 2, \cdots, m_2$. If A and B are both $F(n;\lambda)$ squares, then A and B are orthogonal if when superimposed each combination of elements occurs λ^2 times.

The F squares are generalizations of Latin squares, the Latin squares being $F(n;1)$ F squares. As such, investigations into the structure of F squares prove useful in the study of Latin squares and other related combinatorial systems. If the rows, columns, and elements of the F squares are interpreted as levels of various factors, the F squares provide a valuable source of practical experimental designs. The increased flexibility of the F square over Latin square designs make them substantially more useful. The reader is referred to Hedayat [1969] and

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Hedayat and Seiden [1970] for a more complete discussion of the basic properties and applications of F square designs.

An upper bound on the number of orthogonal $F(n;\lambda)$ squares is $t \le (n-1)^2/(m-1)$, where $m = n/\lambda$. If m is a prime or prime power, $\lambda = m^h$ and $n = m^{h+1}$, a construction has been given for the maximum number $t = (m^{h+1} - 1)^2/(m - 1)$ of orthogonal F squares by Hedayat, Raghavarao, and Seiden [1973]. For composite values of n, little is known of the existence or construction of orthogonal sets. The first case $n = 6 = 2 \cdot 3$ is of particular interest since there are no orthogonal 6×6 Latin squares. Hedayat et al. [1973] exhibit a set of four orthogonal $F(6;2)$ squares, thus demonstrating the existence of at least some nontrivial $F(6;2)$ systems.

The primary objective of this paper is the construction of $F(6;2)$ orthogonal F squares. In section 2 a general construction is presented for sets of orthogonal $F(n,2)$ squares; it was motivated by Anderson [1972]. This construction applied to $n = 6$ produces a set of 8 squares, the maximal set known to date. It is shown that this set of 8 cannot be embedded in a larger set, that is, they are orthogonally locked.

A second construction is presented in section 3 which produces a set of five orthogonal $F(6;2)$ squares. Four of this set may be embedded into a set of eight. The relationships between the sets obtained in sections 2 and 3 are demonstrated.

A third general construction is presented in section 4 for $F(n;2)$ squares where $n + 1 = 4t + 3$ is a prime. For $n = 6$ this construction produces two sets of three pairwise orthogonal $F(6;2)$ squares. For larger n the method produces sequences of F squares for which a square F_i is orthogonal to F_{i-1} and F_{i+1} in the sequence, but not to the remaining.

2. A Method of Differences

<u>Theorem 2.1.</u> <u>Let</u> $n = 2t + 1$. <u>A set of</u> r <u>permutations of integers</u> $-t$ <u>through</u> t <u>produces</u> $r - 1$ <u>orthogonal</u> $F(n+1,2)$ <u>squares if, when placed in an</u> $r \times n$ <u>array,</u>

(a) <u>differences with the first row mod</u> $(n+1)$ <u>reproduce</u> $-t$ <u>through</u> t, <u>and</u>

(b) <u>differences of any other pair mod</u> $(t+1)$ <u>produce</u> 0 <u>one time and</u> $1, 2, \cdots, t$ <u>each two times.</u>

Before giving the proof of this result we illustrate the construction of four orthogonal $F(6;2)$ squares by this method. Consider the following five permutations of integers $-2, -1, 0, 1, 2$.

Permutation					
0 :	-2	-1	0	1	2
1 :	0	-2	1	-1	2
2 :	-1	-2	2	1	0
3 :	-2	0	2	-1	1
4 :	2	-1	1	0	-2

(2.1)

It can be checked that differences of any pair of the first two rows mod(6) reproduce $\{-2, -1, 0, 1, 2\}$. Differences with the last row are, respectively, $\{4, 0, 1, -1, -4\}$, $\{2, 1, 0, 1, -4\}$, $\{3, 1, -1, -1, 2\}$, and $\{4, -1, -1, 1, -3\}$. Reducing the first set mod(6) and the last three mod(3) we observe that the conditions of theorem 2.1 are satisfied.

The F squares are obtained from (2.1) by developing mod(6) the array

$$\begin{array}{lc|ccccc|c|l}
\text{Row} & : & k & k & k & k & k & k & \\
\text{Column} & : & k-2 & k-1 & k & k+1 & k+2 & k+3 & \\
\text{Tr } 1 & : & k & k-2 & k+1 & k-1 & k+2 & k & \\
\text{Tr } 2 & : & k-1 & k-2 & k+2 & k+1 & k & k & \bmod(6). \qquad (2.2)\\
\text{Tr } 3 & : & k-2 & k & k+2 & k-1 & k+1 & k & \\
\text{Tr } 4 & : & k+2 & k-1 & k+1 & k & k-2 & k &
\end{array}$$

The elements of the last four rows corresponding to treatments are then reduced mod(t+1) to produce the F squares. In more typical form the squares are given as

Square 1	Square 2	Square 3	Square 4
1 2 2 0 0 1	2 1 0 0 2 1	2 2 1 0 1 0	1 0 1 0 2 2
2 2 0 0 1 1	2 0 2 1 1 0	1 0 0 2 1 2	0 2 1 2 1 0
2 0 0 1 1 2	1 0 1 0 2 2	0 2 1 1 0 2	1 1 0 2 0 2
0 0 1 1 2 2	0 2 1 2 1 0	0 1 0 2 2 1	0 2 2 1 0 1
0 1 1 2 2 0	1 1 0 2 0 2	2 1 2 1 0 0	2 1 0 0 2 1
1 1 2 2 0 0	0 2 2 1 0 1	1 0 2 0 2 1	2 0 2 1 1 0

(2.3)

We now proceed to the proof of theorem 2.1.

Proof: From the r permutations of integers -t through t create an $r + 1 \times n + 1$ array by

1. letting the first row be $k, k, \cdots, k$,
2. adding k to each of the elements of the r permutations, and (2.4)
3. adjoining the column $(k, k + t + 1, k, \cdots, k)'$,

as in (2.2). The first row is to be a row index, the second (or first row of permutations) is to designate the column index, and the remaining r - 1 rows indicate treatment levels. Develop the array mod(n+1) and reduce the treatment levels mod(t+1). Since each row of the array is a permutation of

$$k + i, \quad i = -t, -t + 1, \cdots, t \tag{2.5}$$

and an additional k, it follows that reduced mod(t+1) each of $\{0, 1, \cdots, t\}$ occurs twice in each row. From condition (a) we note that as k goes from 0 through n + 1 and for any fixed column index the treatment indices mod(n+1) will be a permutation of integers $\{0, 1, \cdots, n\}$. Hence reduced mod(t+1), each of $\{0, 1, \cdots, t\}$ will occur twice in each row and column and the squares are F squares. Condition (b) insures that the squares are orthogonal since as k goes from 0 to n we obtain for each treatment level of the i^{th} treatment, exactly two sets of two for each level of each other factor. The proof is complete.

A second set of r - 1 orthogonal F squares can be obtained by reversing the role of row and column. Let the first permutation of theorem 2.1 reduced mod(n+1) denote rows of the squares. The first column of the i^{th} array, $i = 1, 2, \cdots, r - 1$, except for the t + 1 row, is the $(i + 1)^{st}$ permutation in corresponding order. Let the element in the first column $(t + 1)^{st}$ row be 0. Develop the remaining columns by successively adding one mod(n+1) to each element of the first column. It follows, by reasoning similar to theorem 2.1, that this construction produces a set of r - 1 orthogonal F(n+1,2) squares.

While the second set of F squares are obtained from the same set of permutations, they are not isomorphic. In fact, for n + 1 = 6 the two constructions produce eight pairwise orthogonal F(6,2) squares. We present the F squares to illustrate the construction. It may be checked that they are orthogonal.

Square 5	Square 6	Square 7	Square 8
1 2 0 1 2 0	2 0 1 2 0 1	2 0 1 2 0 1	1 2 0 1 2 0
2 0 1 2 0 1	1 2 0 1 2 0	2 0 1 2 0 1	0 1 2 0 1 2
2 0 1 2 0 1	0 1 2 0 1 2	1 2 0 1 2 0	1 2 0 1 2 0
0 1 2 0 1 2	0 1 2 0 1 2	0 1 2 0 1 2	0 1 2 0 1 2
0 1 2 0 1 2	2 0 1 2 0 1	1 2 0 1 2 0	2 0 1 2 0 1
1 2 0 1 2 0	1 2 0 1 2 0	0 1 2 0 1 2	2 0 1 2 0 1

In section 5 it is demonstrated that this set of eight squares cannot be extended to a larger set.

3. Balanced Incomplete Block Design

A second construction of F(6,2) begins with a simple cyclic array from the 3×3 Latin square.

Row #

$\boxed{1}$	0	0	1	1	2	2	
$\boxed{2}$	0	0	1	1	2	2	
$\boxed{3}$	1	1	2	2	0	0	
$\boxed{4}$	1	1	2	2	0	0	(3.1)
$\boxed{5}$	2	2	0	0	1	1	
$\boxed{6}$	2	2	0	0	1	1	

Consider the balanced incomplete block design with parameters (6, 15, 5, 2, 1) as presented below:

	Rep I	Rep II	Rep III	Rep IV	Rep V	
	1 2	1 3	1 5	1 6	1 4	
Blocks	3 4	2 6	2 3	2 4	2 5	(3.2)
	5 6	4 5	4 6	3 5	3 6	

The levels 0, 1, 2 in the first column of (3.1) partition the row numbers into blocks corresponding to Rep I. We produce the following first columns corresponding to each of the five replications:

Rep I	Rep II	Rep III	Rep IV	Rep V	
0	0	0	0	0	
0	1	2	2	1	
1	0	2	1	2	
1	2	1	2	0	(3.3)
2	2	0	1	1	
2	1	1	0	2	

It should be noted that the incidence relations between levels in any pair of columns is also in a balanced arrangement. For example, consider Rep I and Rep II.

$$\begin{array}{ccc} & & \text{Rep II} \\ & & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \text{Rep I} & \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \tag{3.4}$$

The five F squares with first columns as (3.3) and rows as in (3.1) form a set of orthogonal F(6;2) squares. These squares are not all distinct from those constructed in section 2. By common row and column permutations the squares I, II, III, and IV can be transformed to squares 5, 6, 7, and 8, respectively. Thus these four can be embedded in a set of eight orthogonal F squares corresponding to squares 1, 2, 3, and 4 with proper row and column permutations. The square V is not orthogonal to these four. Further, when square V is transformed by the same row and column permutations as were applied to I, II, III, and IV, and the resulting F square transformed (rows to diagonals) as $5 \to 1$, $6 \to 2$, $7 \to 3$, and $8 \to 4$, we obtain

$$\begin{array}{cccccc} 1 & 2 & 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 & 2 & 0 \end{array} \tag{3.5}$$

which is not an F square.

4. Difference Composition

An attempt to construct sets of orthogonal F(6;2) squares directly from sets of orthogonal 7×7 Latin squares, O(7,6) sets, produced some interesting results. We present the results for primes of the form $p = 4t + 3$, and then consider the case p = 7 as a special case.

Let p be a prime of the form $4t + 3$. Let $\{L_j : j = 1, 2, \cdots, p - 1\}$ denote the set of orthogonal Latin squares specified by

$$L_j = \begin{bmatrix} 0 & 1 & \cdots & p-1 \\ j & 1+j & \cdots & (p-1) + j \\ 2j & 1+2j & \cdots & (p-1) + 2j \\ \vdots & \vdots & & \vdots \\ (p-1)j & 1 + (p-1)j & \cdots & (p-1) + (p-1)j \end{bmatrix} \tag{4.1}$$

Let F_j be obtained from L_j by

(i) squaring each element,

(ii) in the last (p-1) rows replacing the zero by the element in the first column of that row, and (4.2)

(iii) cutting off the first row and first column.

Thus F_j is a $(p-1) \times (p-1)$ array where the entries are the quadratic residues of GF(p).

<u>Theorem 4.1</u>. F_j <u>is</u> <u>an</u> F(p-1,2) F <u>square</u>, $j = 1, 2, \cdots, p - 1$.

<u>Proof</u>: Each row of L_j contains all elements of GF(p) and hence each quadratic residue (q.r.) appears twice and zero one time. The zero has been replaced by the appropriate q.r. for the row, hence each row contains each of the (p-1) q.r. twice. Consider a row with x the element in the first column. The zero in that row occurs in the column headed by $y = p - x$. But $(p - x)^2 = x^2 \pmod p$ so that the same argument as above suffices for the columns.

<u>Theorem 4.2</u>. F_j <u>is</u> <u>orthogonal</u> <u>to</u> F_i <u>if</u> <u>and</u> <u>only</u> <u>if</u> $i = 2j$ <u>or</u> $j = 2i$.

<u>Proof</u>: Consider squares F_j and F_{2j} constructed from L_j and L_{2j}. The removal of the first row of L_j and L_{2j} results in the removal of all pairs (k,k) k = 0, 1, $\cdots$, p - 1, between the two Latin squares. Likewise, the first column removes pairs (k,2k), k = 0, 1, $\cdots$, p - 1. Hence, in order that F_j and F_{2j} be orthogonal F squares these missing matches must be made up in the projection of first column elements into zero positions. Consider the element kj in the first column of L_j. The zero in this row is in the column corresponding to (-kj) mod(p), thus we know the column into which kj, or rather $(kj)^2$, is projected. The corresponding row of L_{2j} is headed by k(2j) and the corresponding position in L_{2j} is occupied by

$$k(2j) + (-kj) = kj,$$

thus in this direction the (kj,kj) matches are made. We must next consider the effect of the projection of first column elements of L_{2j} into the position occupied by kj in L_j and 0 in L_{2j}. Suppose it is the row corresponding to r(2j) and column -r(2j). Then in L_j

$$rj - r(2j) = kj$$

or

$$r = -k.$$

Thus in F_j, $(kj)^2$ is associated with $(kj)^2$ and $(-2kj)^2$ in the projections. Similarly, $(-kj)^2$ occurs with $(-kj)^2$ and $(2kj)^2$. Thus the required matches are attained. Since L_j and L_{2j} are orthogonal Latin squares, F_j and F_{2j} are orthogonal F(p-1,2) squares. Obviously, by the same proof, F_j is orthogonal to F_i if j = 2i. The converse follows by noting that if neither i = 2j or 2i = j, the required matches are not made and F_j is not orthogonal to F_i. The proof is complete.

For p = 7, we obtain six F(6;2) squares F_1, F_2, $\cdots$, F_6. Squares $\{F_1, F_2, F_4\}$ and $\{F_3, F_5, F_6\}$ form two sets of mutually orthogonal F squares. These sets can be embedded in larger orthogonal sets.

The existence of three pairwise orthogonal $F(p-1,2)$ squares by theorem 4.1 is peculiar to $p = 7$. For $p = 11$, we obtain ten $F(10;2)$ squares $F_1, F_2, \cdots, F_{10}$ for which

$$F_1 \perp F_2 \perp F_4 \perp F_8 \perp F_5 \perp F_{10} \perp F_9 \perp F_7 \perp F_3 \perp F_6 \perp F_1 \tag{4.3}$$

but there are no sets of three pairwise orthogonal sets.

In general, theorem 4.1 produces sets of sequences of $F(p-1,2)$ squares such that adjacent pairs are orthogonal. The length of the sequence is determined by the order of 2 in $GF(p)$.

5. Orthogonal Arrays

A $k \times N$ matrix A with entries from a set of $s \geq 2$ elements is called an orthogonal array of size N, k constraints, s levels, and strength t if every $t \times N$ submatrix of A contains all possible $t \times 1$ column vectors with the same frequency λ. Such an array is denoted by the symbol (N,k,s,t) and λ is called the index of the array. Let n_{ij} denote the number of columns that have j coincidences with the i^{th} column. A necessary condition for an array to be an (N,k,s,t) orthogonal array given by Bose and Bush [1952] is that for each $0 \leq h \leq t$ the following inequalities hold:

$$\sum_{j=0}^{k} n_{ij}\binom{j}{h} = \binom{j}{h}(\lambda s^{t-h} - 1) \quad i = 1, 2, \cdots, N. \tag{5.1}$$

Table 5.1 has the eight $F(6;2)$ squares obtained in section 2 in a 10×36 array where the first two rows of the array correspond to the rows and columns of the F squares. Every mapping of pairs of elements $\{0, 1, \cdots, 5\}$ onto $\{0, 1, 2\}$ in rows 1 and 2 produces a $(36, 10, 3, 2)$ orthogonal array. We will use the conditions of (5.1) to show that it is not possible to add an eleventh row to the array of table 5.1, and hence that the eight orthogonal F squares obtained in section 2 cannot be extended to a larger set. For $t = 2$, $s = 3$, $\lambda = 4$ these conditions become

$$\sum_{j=0}^{k} n_{ij} = 4 \cdot 9 - 1 = 35$$

$$\sum_{j=0}^{k} j n_{ij} = (11)k \tag{5.2}$$

$$\sum_{j=0}^{k} j(j-1) n_{ij} = 3k(k-1).$$

<u>Table 5.1</u>: <u>Orthogonal Array (36, 10, 3, 2)</u>

0 0 0 0 0 0	1 1 1 1 1 1	2 2 2 2 2 2	3 3 3 3 3 3	4 4 4 4 4 4	5 5 5 5 5 5
0 1 2 3 4 5	0 1 2 3 4 5	0 1 2 3 4 5	0 1 2 3 4 5	0 1 2 3 4 5	0 1 2 3 4 5
1 2 0 1 2 0	2 0 1 2 0 1	2 0 1 2 0 1	0 1 2 0 1 2	0 1 2 0 1 2	1 2 0 1 2 0
2 0 1 2 0 1	1 2 0 1 2 0	0 1 2 0 1 2	0 1 2 0 1 2	2 0 1 2 0 1	1 2 0 1 2 0
2 0 1 2 0 1	2 0 1 2 0 1	1 2 0 1 2 0	0 1 2 0 1 2	1 2 0 1 2 0	0 1 2 0 1 2
1 2 0 1 2 0	0 1 2 0 1 2	1 2 0 1 2 0	0 1 2 0 1 2	2 0 1 2 0 1	2 0 1 2 0 1
1 2 2 0 0 1	2 2 0 0 1 1	2 0 0 1 1 2	0 0 1 1 2 2	0 1 1 2 2 0	1 1 2 2 0 0
2 1 0 0 2 1	2 0 2 1 1 0	1 0 1 0 2 2	0 2 1 2 1 0	1 1 0 2 0 2	0 2 2 1 0 1
2 2 1 0 1 0	1 0 0 2 1 2	0 2 1 1 0 2	0 1 0 2 2 1	2 1 2 1 0 0	1 0 2 0 2 1
1 0 1 0 2 2	0 2 1 2 1 0	1 1 0 2 0 2	0 2 2 1 0 1	2 1 0 0 2 1	2 0 2 1 1 0

Suppose we reduce the first two rows of table 5.1 mod(3). Table 5.2 gives the incidence numbers for each of the columns headed by $\begin{smallmatrix}3&3&3&3&3&3\\0&1&2&3&4&5\end{smallmatrix}$, respectively, in table 5.1. The incidence parameters for each of these columns is given by:

$$\begin{array}{ccccc} j & 2 & 3 & 4 & 6 \\ n_{ij} & 9 & 16 & 8 & 2 \end{array} \tag{5.3}$$

Table 5.2: Incidence Matrix for Columns $3_0\ 3_1\ 3_2\ 3_3\ 3_4\ 3_5$ of Table 5.1

2	4	4	6	4	4	3	4	3	3	2	3	3	3	4	3	3	2		2	2	6	2	2	3	2	3	3	4	3	3	3	2	3	3	4
4	2	4	4	6	4	3	3	4	3	3	2	2	3	3	4	3	3	2		2	2	6	2	3	3	2	3	3	4	4	3	3	2	3	3
4	4	2	4	4	6	2	3	3	4	3	3	3	2	3	3	4	3	2	2		2	2	6	4	3	3	2	3	3	3	4	3	3	2	3
6	4	4	2	4	4	3	2	3	3	4	3	3	3	2	3	3	4	6	2	2		2	2	3	4	3	3	2	3	3	3	4	3	3	2
4	6	4	4	2	4	3	3	2	3	3	4	4	3	3	2	3	3	2	6	2	2		2	3	3	4	3	3	2	2	3	3	4	3	3
4	4	6	4	4	2	4	3	3	2	3	3	3	4	3	3	2	3	2	2	6	2	2		2	3	3	4	3	3	3	2	3	3	4	3

Let the eleventh row of the array have a zero in the column with all zeros after reducing mod(3). The row must have eleven additional zeros. Let x, y, z, w denote the number of incidence changes from 2 to 3, 3 to 4, 4 to 5, and 6 to 7, respectively. The new incidence relations are then:

j	n_{ij}
2	$9 - x$
3	$16 + x - y$
4	$8 + y - z$
5	z
6	$2 - w$
7	w

(5.4)

Applying the conditions of (5.2) we obtain

$$x + y + z + w = 11$$
$$2x + 3y + 4z + 6w = 30. \tag{5.5}$$

Equations (5.5) have only five possible solutions:

x	y	z	w
6	4	0	1
6	2	3	0
5	4	2	0
4	6	1	0
3	8	0	0

(5.6)

It now remains to consider the addition of an eleventh row to table 5.1 corresponding to one of the solution sets (5.6) which makes an F square, and which is orthogonal to the first ten rows. By exhaustive search it can be demonstrated that none of the solution sets can produce an eleventh row which is an F square orthogonal to the first eight. We thus conclude that this set of eight squares cannot be extended to a larger set. Whether or not there exists a larger set is not known.

Acknowledgment

This work was partially supported by Public Health Research Grant 5-R01-GM-05900, National Institutes of Health.

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